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COMPUTATION OF FLOW OF VISCOELASTIC FLUIDS BY PARAMETER DIFFERENTIATION

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SUMMARY

A technique combining the features of parameter differentiation and finite differences is presented to compute the flow of viscoelastic fluids. Two flow problems are considered: (i) three-dimensional flow near a stagnation point and (ii) axisymmetric flow due to stretching of a sheet. Both flows are characterized by a boundary value problem in which the order of the differential equation exceeds the number of boundary conditions. The exact numerical solutions are obtained using the technique described in the paper. Also, the first-order perturbation solutions (in terms of the viscoelastic fluid parameter) are derived. A comparison of the results shows that the perturbation method is inadequate in predicting some of the vital characteristic features of the flows, which can possibly be revealed only by the exact numerical solution.

KEY WORDS Viscoelastic fluids Parameter differentiation Stagnation point flow Stretching of sheet Finite-difference method Generalized perturbation method

1. INTRODUCTION

Non-Newtonian fluids are increasingly being recognized as more appropriate in modern technological applications in comparison with Newtonian fluids. Because of the non-linear nature of the dependence of stresses on the rate of strain for non-Newtonian fluids, the solutions of flow problems for these fluids are in general more difficult to obtain. This is not only true of exact analytical solutions but even of numerical solutions.

Of particular interest is a class of fluids which have come to be known as viscoelastic fluids in the literature. The constitutive equation of this class of fluids is relatively simple, yet for such fluids considerable difficulties have been encountered in obtaining accurate numerical solutions of various flow problems. For over the last two decades, their solutions have been reported in the literature making simplifying assumptions which have been of late put under a cloud by recent researches.

Consider, for example, the two-dimensional flow of a particular viscoelastic fluid, namely Walters B′ fluid, near a stagnation point. The boundary value problem (BVP) governing the motion is

$$f'''+ff''+1-f'^2+k(ff^{\mathrm{iv}}-2f'f'''+f''^2)=0, \tag{1}$$

with the boundary conditions

$$f(0)=0, \qquad f'(0)=0, \qquad \lim_{\eta\to\infty} f'(\eta)=1, \tag{2}$$

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where f is a non-dimensional measure of the streamfunction, k is the non-Newtonian fluid parameter and a prime denotes the derivative with respect to η, the similarity variable.

The difficulties in obtaining a numerical solution of the above BVP stem from the fact that the introduction of the non-Newtonian parameter increases the order of the differential equation from three to four, without any corresponding extra boundary condition. Beard and Walters[1] resolved these difficulties by assuming

$$f = f_0 + k f_1, \tag{3}$$

where f_0 is the solution for the Newtonian fluid and f_1 is the perturbation in the solution due to the non-Newtonian nature of the fluid. It turns out that the BVPs for both f_0 and f_1 are of third order, with three boundary conditions, and their solutions can be obtained by any of the standard techniques such as the shooting method, finite differences, quasi-linearization or invariant imbedding. There is another attractive feature of this method: the solution curves for f_0 and f_1 being universal, the solution for an arbitrary value of k (small) can be obtained by simply superimposing the solutions of f_0 and f_1.

The literature abounds in the study of flow problems characterized by similar BVPs. Since the earlier effort of Beard and Walters,[1] numerous researchers (see e.g. References 2–5) have likewise tried to resolve the problems arising out of increased order of differential equations due to viscoelasticity of the fluid by seeking a perturbation solution of the type (3). They have sought refuge in the argument: 'Implicit in the derivation of equation (1) is the neglect of terms of order k^2. It is therefore reasonable to seek a solution of BVP (1), (2) of the form (3)'. That such an argument can lead to erroneous results was first shown by Frater,[6] who, considering the problem of steady flow parallel to an infinite porous plate, demonstrated that the use of assumption (3) can lead to an overshoot of the velocity over its value in the mainstream. This example naturally shed doubts on the validity of the results obtained in Reference 1 and also implicitly on the results obtained in the vast body of literature on similar problems.[2–5]

Under the impression that numerical solutions of BVP (1), (2) could not possibly be obtained because of the lack of additional boundary condition, attempts were made to obtain its solution by other techniques, such as von Karman–Paulhausen integral methods[7,8] and weighted residual methods.[9,10] None of these investigations shed any light on the question of the velocity overshooting its value in the mainstream.

Recently two attempts were made independently by Teipel[11] and the present author[12] to obtain the exact numerical solution of BVP (1), (2). Both these investigations have demonstrated that for a bounded solution no extra boundary condition is required. This is because f vanishes at $\eta = 0$. Teipel[11] used a Taylor series expansion for f near $\eta = 0$ to obtain a solution for f up to $\eta = 0{\cdot}1$ and from that point on used the Runge–Kutta method to get the solution for the remaining range of η. The present author,[12] on the other hand, presented a method which is uniformly applicable for all values of k, including $k \to 0$ and $k = 0$. It combines the features of finite differences and the shooting method. Whereas Teipel's method is likely to be unstable for vanishingly small values of k, the method proposed in Reference 12 is free from this drawback. Both these investigations have reported the oscillatory nature of the boundary layer, resulting in the overshooting of the velocity in the boundary layer over its value in the mainstream, and the non-admissibility of the solutions for values of k exceeding some critical value. Teipel[11] erroneously asssumed that the shear stress would become infinite for these values of k. The present author,[12] however, demonstrated that the solutions could not be obtained because there is a turning point in the solution. Both of these authors have, nevertheless, underscored an important principle, namely that for viscoelastic fluids the solutions obtained by seeking a perturbation solution of the form (3) are *not* guaranteed to give acceptable results, qualitatively or quantitatively, and that *there is no alternative to seeking exact solutions (analytical or numerical) of the original set of equations characterizing the flow of viscoelastic fluids.* The present paper, emphasizing this principle, is one of several currently under preparation by the present author.

In the present paper we consider two problems relating to three-dimensional axisymmetric flow of Walters B′ fluid: (i) three-dimensional flow near a stagnation point and (ii) three-dimensional flow due to stretching of a sheet radially. The equations of motion for both problems are similar to equation (1). One could therefore apply the method enunciated in Reference 12. However, for a two-dimensional stagnation point flow a turning point was discovered in the solution. Anticipating a similar phenomenon for the three-dimensional flow, in the present paper we seek a *family* of solutions by steadily increasing the value of $f''(0)$ and determining the corresponding value of k. The solution proposed in Reference 12 is iterative. The iterations can be avoided by appealing to the idea of parameter differentiation.[13] This has been done in the present paper and the solutions of the resulting linear differential equation have been obtained by incorporating the ideas of Reference 12.

For the second problem, even though one does not expect a turning point in the solution, a family of solutions is sought by varying the value of k and determining $f''(0)$. Once again the ideas of parameter differentiation and those in Reference 12 are combined to obtain the solution for a range of values of k for which the solution exists.

2. EQUATIONS OF MOTION

The equations of motion for Walters B′ fluid are[14]

$$\rho\left(\frac{\partial \mathbf{v}}{\partial t}+(\mathbf{v}\cdot\nabla)\mathbf{v}\right)=-\nabla p+\eta_0\nabla^2\mathbf{v}-k_0\left(\frac{\partial}{\partial t}\nabla^2\mathbf{v}+2(\mathbf{v}\cdot\nabla)\nabla^2\mathbf{v}-\nabla^2[(\mathbf{v}\cdot\nabla)\mathbf{v}]\right), \tag{4}$$

$$\nabla\cdot\mathbf{v}=0, \tag{5}$$

where ρ, p, $\mathbf{v}$, η_0 and k_0 denote respectively the density, pressure, velocity, limiting viscosity at small rate of shear and the short-memory coefficient of the fluid at a point.

In the present paper we are concerned with steady axisymmetric flow of the fluid; therefore

$$\partial/\partial t=0, \qquad \partial/\partial\theta=0 \tag{6}$$

and the velocity field has the components $(u, 0, w)$ in the cylindrical co-ordinate system (r, θ, z).

The equations of motion (4) and (5) thus become

$$\rho\left(u\frac{\partial u}{\partial r}+w\frac{\partial u}{\partial z}\right)=-\frac{\partial p}{\partial r}+\eta_0\nabla'^2u-k_0\left[2\left(u\frac{\partial}{\partial r}+w\frac{\partial}{\partial z}\right)\nabla'^2u-\nabla'^2\left(u\frac{\partial u}{\partial r}+w\frac{\partial u}{\partial z}\right)\right], \tag{7}$$

$$\rho\left(u\frac{\partial w}{\partial r}+w\frac{\partial w}{\partial z}\right)=-\frac{\partial p}{\partial z}+\eta_0\nabla^2w-k_0\left[2\left(u\frac{\partial}{\partial r}+w\frac{\partial}{\partial z}\right)\nabla^2w-\nabla^2\left(u\frac{\partial w}{\partial r}+w\frac{\partial w}{\partial z}\right)\right], \tag{8}$$

$$\frac{\partial u}{\partial r}+\frac{u}{r}+\frac{\partial w}{\partial z}=0, \tag{9}$$

where

$$\nabla'^2=\nabla^2-1/r^2. \tag{10}$$

For the two problems under consideration, namely the three-dimensional flow near a stagnation point and the axisymmetric flow due to radial stretching of a sheet, we look for a similarity solution in which the velocity components are given by

$$u = rf'(z), \qquad w = -2f(z), \tag{11}$$

where a prime denotes the derivative with respect to z.

Note that the choice (11) of u and w automatically satisfies the continuity equation (9). If we substitute for u and w in equations (7) and (8), we obtain

$$\rho(f'^2 - 2ff'') = -\frac{1}{r}\frac{\partial p}{\partial r} + \eta_0 f''' + 2k_0(ff^{\text{iv}} - 2f'f'''), \tag{12}$$

$$4\rho ff' = -\frac{\partial p}{\partial z} - 2\eta_0 f'' + 4k_0(3f'f'' - ff'''). \tag{13}$$

Integration of equation (13) with respect to z yields

$$p = g(r) - 2\rho f^2 - 2\eta_0 f' + 4k_0(2f'^2 - ff''), \tag{14}$$

where $g(r)$ is an arbitrary function of r.

Differentiating equation (14) partially with respect to r, we obtain

$$\frac{1}{r}\frac{\partial p}{\partial r} = \frac{g'(r)}{r}. \tag{15}$$

Since in equation (12) $(1/r)\ \partial p/\partial r$ is a function of r only by virtue of equation (15) and the remaining terms are functions of z only, it follows that $(1/r)\ \partial p/\partial r$ must be a constant. Let

$$\frac{1}{r}\frac{\partial p}{\partial r} = \frac{g'(r)}{r} = A. \tag{16}$$

Integration of equation (16) yields

$$g(r) = \tfrac{1}{2}Ar^2 + B, \tag{17}$$

A and B being constants.

Hence the pressure p at any point is given by

$$p = \tfrac{1}{2}Ar^2 + B - 2\rho f^2 - 2\eta_0 f' + 4k_0(2f'^2 - ff''). \tag{18}$$

The values of the constants A and B will be determined by matching the pressure in the mainstream flow.

Substituting for $(1/r)\ \partial p/\partial r$ from equation (16) in equation (12), we obtain

$$\rho(f'^2 - 2ff'') = -A + \eta_0 f''' + 2k_0(ff^{\text{iv}} - 2f'f'''). \tag{19}$$

Since the value of A depends on the mainstream flow, at this point we shall find it convenient to deal with the two problems at hand separately.

3. THREE-DIMENSIONAL FLOW NEAR A STAGNATION POINT

For the three-dimensional flow near a stagnation point the mainstream flow is the potential flow in which the velocity components are given by

$$u_\infty = Er, \qquad w_\infty = -2Ez, \tag{20}$$

E being a constant.

Using Bernaulli's equation, the pressure in the potential flow is given by

$$p_\infty = p_0 - \tfrac{1}{2}\rho(u_\infty^2 + w_\infty^2), \tag{21}$$

where p_0 is the pressure at the stagnation point.

Substitution of u_∞ and w_∞ from equation (20) yields

$$p_\infty = p_0 - \tfrac{1}{2}\rho E^2(r^2 + 4z^2). \tag{22}$$

Matching of p in equations (18) and (22) gives

$$A = -\rho E^2, \qquad B = p_0. \tag{23}$$

Thus we have the following expression for pressure:

$$p = p_0 - \tfrac{1}{2}\rho E^2 r^2 - 2\rho f^2 - 2\eta_0 f' + 4k_0(2f'^2 - ff''). \tag{24}$$

If we substitute for A from equation (23) into equation (19), we obtain the following differential equation for f:

$$\rho(f'^2 - 2ff'') = \rho E^2 + \eta_0 f''' + 2k_0(ff^{\text{iv}} - 2f'f'''). \tag{25}$$

Equation (25) can be non-dimensionalized by introducing the variables

$$\zeta = \left(\frac{E\rho}{\eta_0}\right)^{1/2} z, \qquad f(z) = \left(\frac{E\eta_0}{\rho}\right)^{1/2} \phi(\zeta), \tag{26}$$

in which case equation (25) takes the form

$$\phi''' + 2\phi\phi'' + 1 - \phi'^2 + 2k(\phi\phi^{\text{iv}} - 2\phi'\phi''') = 0, \tag{27}$$

where

$$k = k_0 E/\eta_0 \tag{28}$$

is the dimensionless measure of the viscoelasticity of the fluid and a prime *now* denotes the derivative with respect to ζ rather than with respect to z.

The boundary conditions on ϕ correspond to the no-slip condition at the surface $z=0$ and the asymptotic limit to the mainstream flow as $z \to \infty$ and are given by

$$\phi(0) = 0, \qquad \phi'(0) = 0, \qquad \phi'(\infty) = 1. \tag{29}$$

3.1. *Parameter differentiation*

It has been shown by Teipel[11] and Ariel[12] in an analogous situation that, notwithstanding the fact that the order of the differential equation governing ϕ is four and there are only three boundary conditions on ϕ, if $\phi''(0)$ is known that satisfies the asymptotic boundary condition $\phi'(\infty) = 1$, then the solution for ϕ can be obtained. Let us then assume that

$$\phi''(0) = s. \tag{30}$$

When $k=0$, i.e., for a Newtonian fluid, the flow field is well known. It was given by Homman,[15] who found numerically

$$s_0 = \phi''(0) = 1{\cdot}312.$$

We shall now increase the value of s gradually from its value 1·312 for $k=0$ and determine the corresponding value of k. Differentiating equation (27) with respect to s, we obtain

$$F''' + 2\phi F'' - 2\phi'F' + 2\phi''F + 2k(\phi F^{\text{iv}} - 2\phi'F''' - 2\phi'''F' + \phi^{\text{iv}}F) + 2\beta(\phi\phi^{\text{iv}} - 2\phi'\phi''') = 0, \tag{31}$$

where

$$F = \frac{\partial \phi}{\partial s}, \qquad \beta = \frac{\partial k}{\partial s}. \tag{32}$$

The boundary conditions on F are

$$F(0)=0, \qquad F'(0)=0, \qquad F''(0)=1, \qquad F'(\infty)=0. \tag{33}$$

Note that equation (31) is a linear differential equation. Therefore its solution can be found non-iteratively. One can indeed use the principle of superimposition to obtain the value of β. We write

$$F = G + \beta H, \tag{34}$$

where G and H satisfy

$$\begin{gathered} G''' + 2\phi G'' - 2\phi' G' + 2\phi'' G + 2k(\phi G^{\text{iv}} - 2\phi' G''' - 2\phi''' G' + \phi^{\text{iv}} G) = 0, \\ G(0)=0, \qquad G'(0)=0, \qquad G''(0)=1, \end{gathered} \tag{35}$$

$$\begin{gathered} H''' + 2\phi H'' - 2\phi' H' + 2\phi'' H + 2k(\phi H^{\text{iv}} - 2\phi' H''' - 2\phi''' H' + \phi^{\text{iv}} H) + 2(\phi\phi^{\text{iv}} - 2\phi'\phi''') = 0, \\ H(0)=0, \qquad H'(0)=0, \qquad H''(0)=0. \end{gathered} \tag{36}$$

The value of β can now be found by using the terminal condition $F'(\infty)=0$. We have

$$\beta = -G'(\infty)/H'(\infty). \tag{37}$$

It may be noted that the sets of equations (35) and (36) are initial value problems (IVPs). Thus one marches forward with the given initial conditions and carries out integration until β, given by equation (37), becomes stationary. This is the required value of β. When substituted in equation (34), it will give the value of F.

Once the values of F and β are known, which respectively denote the rate of increase of ϕ and k with respect to s, the values of ϕ and k can be obtained at the next step of s, in principle at any rate. Hence the solution can be developed up to any desired value of s by continuously increasing its value.

The IVPs (35) and (36) can be solved by following either the technique of Teipel[11] or that given by Ariel.[12] In the present paper we follow the latter technique since it allows the solution to be developed for all values of k, including $k=0$ and $k\to 0$.

3.2. *Discretization of IVPs*

Solution for G: We write

$$\begin{gathered} \phi = y_1, \qquad \phi' = y_2, \qquad \phi'' = y_3, \\ G = z_1, \qquad G' = z_2, \qquad G'' = z_3. \end{gathered} \tag{38}$$

The IVP (35) can be rewritten as

$$\begin{gathered} z_1' = z_2, \qquad z_2' = z_3, \\ z_3' + 2y_1 z_3 - 2y_2 z_2 + 2y_3 z_1 + 2k(y_1 z_3'' - 2y_2 z_3' - 2y_3' z_2 + y_3'' z_1) = 0, \\ z_1(0)=0, \qquad z_2(0)=0, \qquad z_3(0)=1. \end{gathered} \tag{39}$$

Introducing a mesh defined by

$$\zeta_i = ih, \quad i = 0, 1, 2, \ldots, N, \tag{40}$$

where h is the mesh size and N is a sufficiently large number, one can discretize the above system to obtain

$$\frac{z_3^{j+1} - z_3^{j-1}}{2h} + 2y_1^j z_3^j - 2y_2^j z_2^j + 2y_3^j z_1^j$$
$$+ 2k\left(y_1^j \frac{z_3^{j+1} - 2z_3^j + z_3^{j-1}}{h^2} - 2y_2^j \frac{z_3^{j+1} - z_3^{j-1}}{2h} - 2\frac{y_3^{j+1} - y_3^{j-1}}{2h} z_2^j + \frac{y_3^{j+1} - 2y_3^j + y_3^{j-1}}{h^2} z_1^j \right) = 0, \tag{41}$$

$$z_2^{j+1} = z_2^j + \tfrac{1}{2}h(z_3^j + z_3^{j+1}), \tag{42}$$

$$z_1^{j+1} = z_1^j + \tfrac{1}{2}h(z_2^j + z_2^{j+1}), \tag{43}$$

$$z_1^0 = 0, \qquad z_2^0 = 0, \qquad z_3^0 = 1. \tag{44}$$

Note that since we are using central finite differences for the derivatives and averages, the accuracy of the above system is $O(h^2)$.

Equation (41) can be solved explicitly for z_3^{j+1} to yield

$$z_3^{j+1} = \left[1 + 4k\left(\frac{y_1^j}{h} - y_2^j \right) \right]^{-1} \left[z_3^{j-1} - 4h(y_1^j z_3^j - y_2^j z_2^j + y_3^j z_1^j) \right.$$
$$\left. - 4k\left(y_1^j \frac{-2z_3^j + z_3^{j-1}}{h} + y_2^j z_3^{j-1} - z_2^j (y_3^{j+1} - y_3^{j-1}) + z_1^j \frac{y_3^{j+1} - 2y_3^j + y_3^{j-1}}{h} \right) \right]. \tag{45}$$

Equation (45) involves the values of z_3 at three adjacent mesh points, namely $j-1$, j and $j+1$. Since only the value of z_3^0 is known, in order to use equation (45) iteratively, one needs to know the value of z_3^1. This can be found by seeking a Taylor series expansion of z_3 or G'' around $\zeta = 0$. We have

$$z_3^1 = G''(h) = G''(0) + hG'''(0) + \frac{h^2}{2!} G^{\text{iv}}(0) + O(h^3) = 1 - 2h^2 k \tag{46}$$

within an accuracy $O(h^2)$.

The value of $G^{\text{iv}}(0)$ can be obtained by differentiating the differential equation (35) and setting $\zeta = 0$.

Now, since z_3^1 is known, making use of the initial conditions (44), z_2^1 and z_1^1 can be found directly from equations (42) and (43) respectively. Next z_3^2 is determined from equation (45) and z_2^2 and z_1^2 from equations (42) and (43) respectively. The cycle is repeated until the values of the vector $\mathbf{z} = (z_1, z_2, z_3)$ are determined at all mesh points. It is worth noting that the values of the components of $\mathbf{z}$ must be calculated in a specific manner. First z_3, then z_2 and finally z_1 are computed using equations (45), (42) and (43) respectively.

Solution for H. We can similarly write

$$\phi = y_1, \qquad \phi' = y_2, \qquad \phi'' = y_3,$$
$$H = z_1, \qquad H' = z_2, \qquad H'' = z_3. \tag{47}$$

IVP (36) can be restated as

$$
\begin{gathered}
z_1' = z_2, \qquad z_2' = z_3, \\
z_3' + 2y_1 z_3 - 2y_2 z_2 + 2y_3 z_1 + 2k(y_1 z_3'' - 2y_2 z_3' - 2y_3' z_2 + y_3'' z_1) + 2(y_1 y_3'' - 2y_2 y_3') = 0, \\
z_1(0) = 0, \qquad z_2(0) = 0, \qquad z_3(0) = 0.
\end{gathered} \tag{48}
$$

It may be noted that the only differences between the above system of equations and system (39) are in the presence of the non-homogeneous term $2(y_1 y_3'' - 2y_2 y_3')$ in the differential equation for z_3 and the null initial value of z_3. Equation (45) is therefore replaced by

$$
\begin{aligned}
z_3^{j+1} = & \left[1 + 4k\left(\frac{y_1^j}{h} - y_2^j\right)\right]^{-1}\Bigg[z_3^{j-1} - 4h(y_1^j z_3^j - y_2^j z_2^j + y_3^j z_1^j) \\
& -4k\left(y_1^j \frac{-2z_3^j + z_3^{j-1}}{h} + y_2^j z_3^{j-1} - z_2^j (y_3^{j+1} - y_3^{j-1}) + z_1^j \frac{y_3^{j+1} - 2y_3^j + y_3^{j-1}}{h}\right) \\
& -4h\left(y_1^j \frac{y_3^{j+1} - 2y_3^j + y_3^{j-1}}{h} - y_2^j (y_3^{j+1} - y_3^{j-1})\right)\Bigg].
\end{aligned} \tag{49}
$$

Also, because of the different initial value of z_3, equation (46) is replaced by

$$
z_3^1 = -2sh^2. \tag{50}
$$

The procedure for computing the values of $\mathbf{z}$ at various mesh points is identical to the one outlined above and will not be repeated here. The computations of G and H need to be carried out simultaneously until the value of β given by equation (37) becomes stationary. At this point F and its derivatives can be obtained from equation (34).

3.3. *Calculation of ϕ at the next value of s*

Since the values of F and β have been found, it should be possible, in principle at any rate, to obtain the values of ϕ and k at the next step of s by integrating equation (32). Na[13] in his celebrated work and other published material (see e.g. Reference 16) recommends the use of the simple Euler method. Thus if $\Delta\phi$ and Δk represent the increments in ϕ and k respectively, corresponding to an increment Δs in s, from equation (32) one can write

$$
\Delta\phi = F\,\Delta s, \qquad \Delta k = \beta\,\Delta s. \tag{51}
$$

The accuracy of the Euler method is $O(\Delta s)$ only. This can be improved substantially by using other integration algorithms (see e.g. Reference 17). However, in order to preserve the simplicity of the method of parameter differentiation, it is desirable to make use of the values of ϕ and k at the integration steps only, and not at 'half-steps' as required in Runge–Kutta methods. Adam–Bashforth methods fit in nicely with the above requirements since they require the values only at the previous integration steps. In the present paper we have used the two-step Adam–Bashforth method, which yields an accuracy $O(\Delta s)^2$ and is consistent with the discretization scheme used in the space domain.

For a two-step Adam–Bashforth method one needs an additional set of values for ϕ and k at the level $s_0 + \Delta s$. Then at the subsequent levels $s_0 + j\Delta s$ ($j > 1$) the solution can be developed inductively. The values at the level $s_0 + \Delta s$ can be obtained within an accuracy $O(\Delta s)^2$ by either differentiating equation (31) once again with respect to s to obtain the values of $\partial^2\phi/\partial s^2$ and

$\partial^2 k/\partial s^2$ and using

$$\phi(s_0+\Delta s)=\phi(s_0)+\frac{\partial\phi}{\partial s}\Delta s+\frac{1}{2}\frac{\partial^2\phi}{\partial s^2}(\Delta s)^2,$$
$$k(s_0+\Delta s)=k(s_0)+\frac{\partial k}{\partial s}\Delta s+\frac{1}{2}\frac{\partial^2 k}{\partial s^2}(\Delta s)^2, \tag{52}$$

or by solving BVP (27), (29) iteratively using a suitable iterative scheme such as the one detailed by the present author in Reference 12. Since the expensive iterations take place only at one level, it is a small premium to pay for the greatly improved accuracy of the solution. In the present paper we have used the iterative scheme[12] to obtain the values of ϕ and k at the first level $(s_0+\Delta s)$. Note that the method proposed by Teipel[11] would not be a suitable choice, since the value of k sought at the first level is rather small.

There was another consideration which warranted attention. The values of k (and ϕ) vary unevenly with s. An inspection of the k–s curve[12] shows the direction of the tangent varying from nearly horizontal to vertical. In order to optimize the computing resources, it is desirable to cater for variable step sizes of s. For this reason the Adam–Bashforth method of second order, appropriate to variable step size, was used.[17] The value of Δs was varied from 0·001 to 0·5 depending on the slope of the k–s curve. The value of ζ where β became stationary also depended strongly on the value of k. Thus for $k\ll 1$, i.e. for a slightly non-Newtonian fluid, β practically became constant (five significant digits) at $\zeta=4$, but for $k=0{\cdot}15$ the same could be realized only at $\zeta=20$. Finally, the accuracy of the results was checked periodically by computing the values of ϕ and k iteratively at selected values of s.

3.4. *Results and discussion*

In Figure 1 the velocity profile $\phi'(\zeta)$ is plotted against ζ for various values of k. As reported by Teipel[11] and Ariel,[12] oscillations occur about the value of the mainstream velocity ($\phi'=1$) for all non-zero values of k, the amplitude of the oscillations increasing with increasing value of k. This paper thus further confirms the original conclusion of Beard and Walters,[1] namely that the viscoelasticity of the fluid causes the velocity in the boundary layer to exceed its value in the mainstream flow. Also, it furnishes evidence that the oscillatory behaviour of the boundary layer for viscoelastic fluids, first reported by Teipel[11] and Ariel,[12] is not due the numerical scheme chosen by those authors but is an inherent physical characteristic of the fluid. That this is indeed the case is further demonstrated in the Appendix, where the solution of BVP (27), (29) is derived using the generalized perturbation technique.

In accordance with the standard practice of parameter differentiation, the value of s $(=\phi''(0))$ was increased from its initial value s_0. The solutions could be obtained only for $k\le k_c$, where

$$k_c=0{\cdot}185\,909.$$

No solutions could be obtained for $k>k_c$.

However, just as in the case of the two-dimensional flow, another branch of the solution was discovered when s was further increased. For this branch the value of k monotonically decreases with an increase in the value of s. These findings are illustrated in Figure 2, in which $\phi''(0)$ is plotted against k. One can notice the turning point in the figure at $k=k_c$. It is also clear that dual solutions exist for all values of $k<k_c$.

In Figure 3 the dual solutions for velocity profile are plotted for some selected values of k near k_c. From the figure it can be seen that the amplitude of the oscillations increases sharply for the

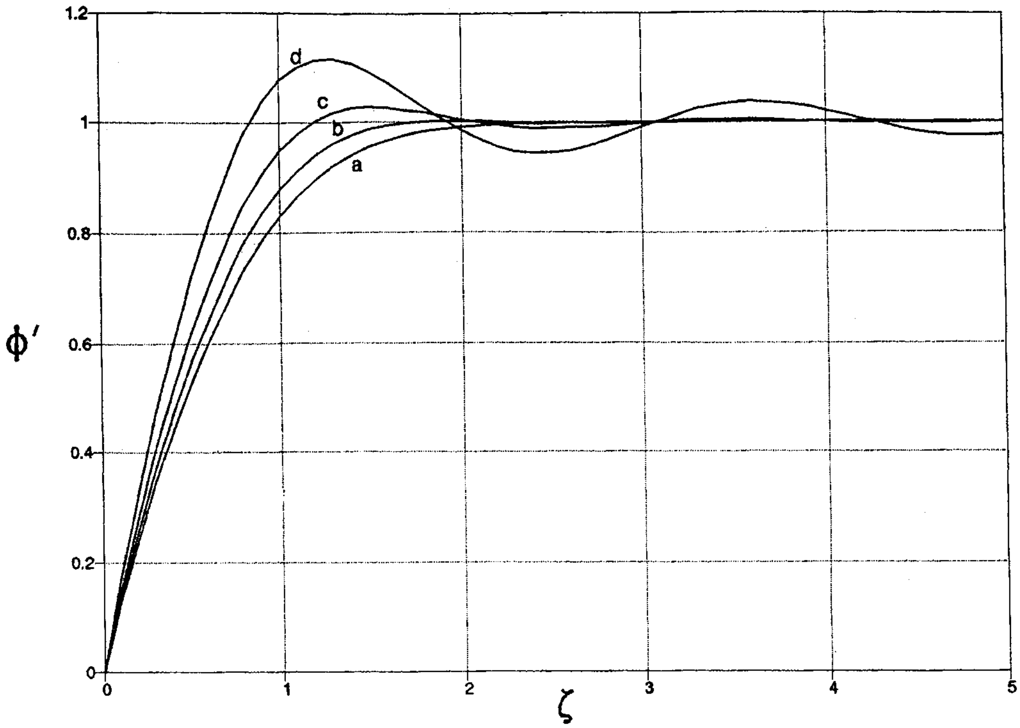

Figure 1. Three-dimensional flow near a stagnation point: variation in velocity distribution $\phi'(\zeta)$ with ζ, the similarity variable, for various values of k, the viscoelastic parameter. Curve a, $k=0{\cdot}00$; curve b, $k=0{\cdot}05$; curve c, $k=0{\cdot}10$; curve d, $k=0{\cdot}15$

upper branch of the solution as k is decreased from its maximum value k_c. For this solution it was found that for values of k less than a certain value the amplitude of the osciallations exceeds unity, giving rise to backward flow ($\phi'<0$) for some values of ζ. A further decrease in the value of k increases the amplitude even more, which strongly suggests that the solutions corresponding to the upper branch in Figure 2 are unstable.

3.5. *Results using perturbation technique*

It may be of some interest to compare the results obtained by the technique used in the present paper with those obtained by using the perturbation technique characterized by equation (3), since the latter has been extensively used in the literature.

Writing

$$\phi=\phi_0+k\phi_1 \tag{53}$$

and expanding equation (27) to the first order of smallness in k, one obtains the following pair of BVPs:

$$\begin{gathered}\phi_0'''+2\phi_0\phi_0''+1-\phi_0'^2=0,\\ \phi_0(0)=0, \qquad \phi_0'(0)=0, \qquad \phi_0'(\infty)=1,\end{gathered} \tag{54}$$

$$\begin{gathered}\phi_1'''+2\phi_0\phi_1''-2\phi_0'\phi_1'+2\phi_0''\phi_1=-2(\phi_0\phi^{\mathrm{iv}}-2\phi_0'\phi_0'''),\\ \phi_1(0)=0, \qquad \phi_1'(0)=0, \qquad \phi_1'(\infty)=0.\end{gathered} \tag{55}$$

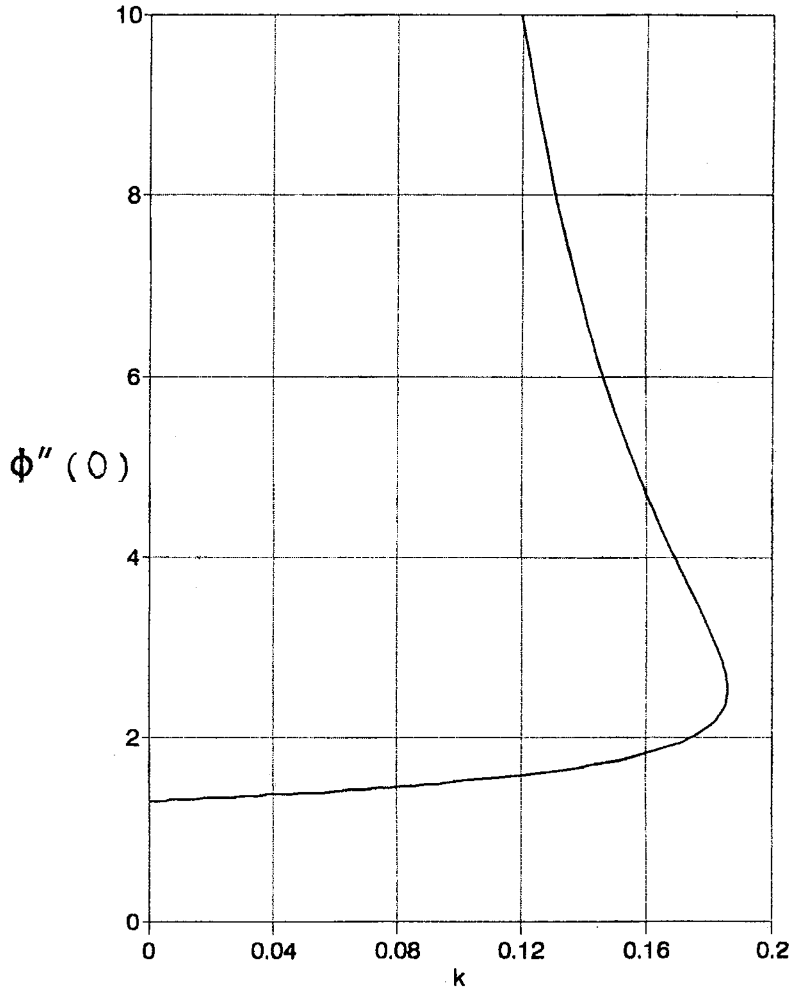

Figure 2. Three-dimensional flow near a stagnation point: variation in $\phi''(0)$ with k, the viscoelastic parameter. There is a turning point at $k=k_c$ ($=0{\cdot}185\,909$). Dual solutions exist for $k<k_c$. No solution exists for $k>k_c$.

BVP (54) is well known as characterizing Homman's flow. Its solution is reproduced in textbooks on fluid dynamics. The solution of BVP (55) can be obtained in a single iteration, since it is a linear BVP, by invoking any of the standard techniques, such as the shooting method. Using this technique, the missing initial condition was found as

$$\phi_1''(0)=1{\cdot}481\,487.$$

In Table I the values of $\phi''(0)$ are listed using the technique of parameter differentiation suggested in the present paper and the perturbation technique described above. One can note the increasing discrepancy between the values as the value of k is increased. In particular, using the perturbation method, the results for $\phi''(0)$ have the maximum error near the turning point where there is a vertical tangent. Of course, beyond $k=k_c$ the perturbation method would still give results, whereas in theory no solution exists for these values!

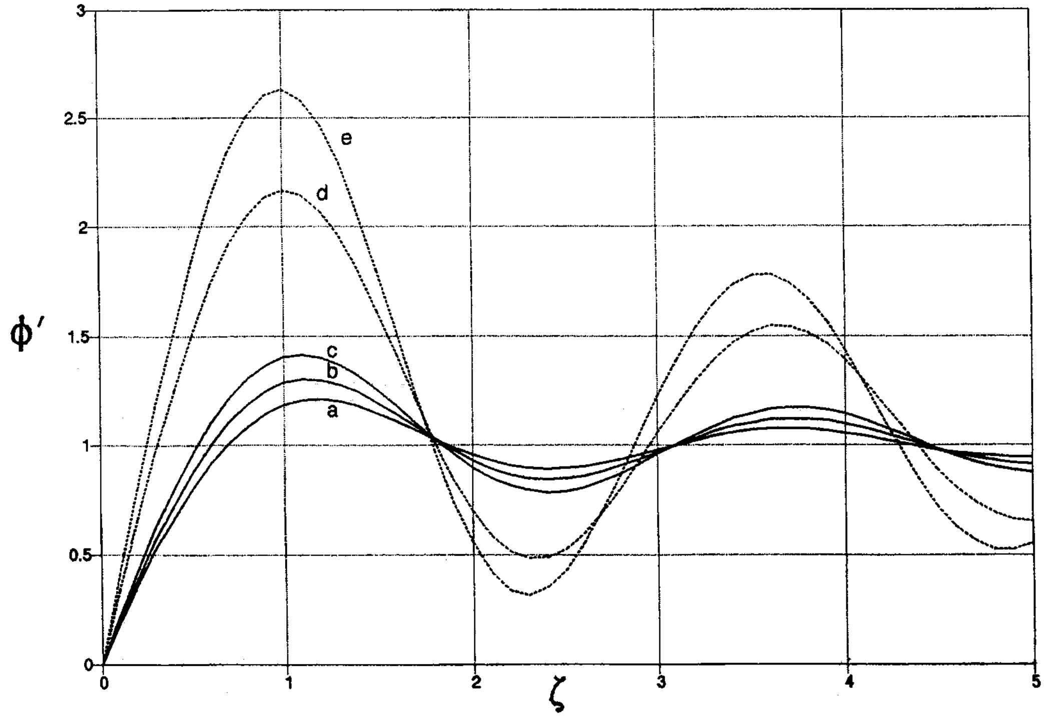

Figure 3. Three-dimensional flow near a stagnation point: dual solutions. The velocity distribution $\phi'(\zeta)$ is plotted against ζ, the similarity variable, for various values of k. The dotted curves indicate the solution on the upper branch of the curve in Figure 2. The solutions corresponding to these curves are likely to be unstable. Curve a, $k=0{\cdot}17$; curve b, $k=0{\cdot}18$; curve c, $k=k_c$; curve d, $k=0{\cdot}18$; curve e, $k=0{\cdot}17$

Table I. Variation in $\phi''(0)$ with k for three-dimensional flow near a stagnation point

k	$\phi''(0)$	
	Exact	Perturbation
0·00	1·311938	1·311938
0·05	1·397741	1·386012
0·10	1·520806	1·460087
0·15	1·739889	1·534161

From the foregoing one can conclude that in order to obtain accurate information about the nature of the solution of flow problems of viscoelastic fluids, the full set of equations must be used rather than the perturbed ones. The present paper furnishes yet another algorithm which obviates the difficulty characterizing the BVP of type (1) and (2), in which the order of the differential equation exceeds the number of boundary conditions. Using this algorithm, it is possible to determine the enitre family of solutions which exist for various values of k and also the turning point which limits the values of k, thus giving the range of k for which solutions are feasible. Finally, using the present algorithm, another branch of solution could be found, even though it appears highly likely that it is unstable. It may be remarked that these conclusions could not possibly be reached using the perturbation technique, even though the values of k under question were quite 'small' ($<0{\cdot}2$, a value used by Beard and Walters[1]).

4. AXISYMMETRIC FLOW DUE TO RADIAL STRETCHING OF SHEET

In this section we turn our attention to the flow of viscoelastic fluid caused by the radial stretching of a sheet. The corresponding two-dimensional flow has been considered by Rajagopal *et al.*,[4] who obtained the perturbation solution for the flow. An exact solution of the same problem was spotted by Troy *et al.*[18] Ariel,[12] using an iterative technique, obtained the numerical solution of the full set of equations and demonstrated that the numerical value of τ, the dimensionless stress on the sheet, was in complete agreement with the exact analytical value. This fact should also give confidence in the numerical results derived in the present paper, since they were checked using the iterative technique reported in Reference 12.

Analogously to the two-dimensional case, we take the radial velocity of the sheet to be proportional to the distance from the stagnation point. The equation of motion, (25), is modified in view of the fact that now $u_\infty = w_\infty = 0$. Consequently, we have the following equation for ϕ:

$$\phi''' + 2\phi\phi'' - \phi'^2 + 2k(\phi\phi^{\text{iv}} - 2\phi'\phi''') = 0. \tag{56}$$

The boundary conditions on ϕ become

$$\phi(0) = 0, \qquad \phi'(0) = 1, \qquad \phi'(\infty) = 0. \tag{57}$$

The BVP defined by equations (56) and (57) is very similar to the one discussed in the previous section and one can use the parameter differentiation technique to solve it. However, for the present problem there is no particular reason to 'invert' the problem and obtain k for a range of values of $\phi''(0)$. One can proceed 'directly' and obtain $\phi''(0)$ for a range of values of k.

4.1. Parameter differentiation

Differentiating equation (52) with respect to k, we obtain

$$F''' + 2\phi F'' - 2\phi' F' + 2\phi'' F + 2k(\phi F^{\text{iv}} - 2\phi' F''' - 2\phi''' F' + \phi^{\text{iv}} F) + 2(\phi\phi^{\text{iv}} - 2\phi'\phi''') = 0, \tag{58}$$

where now

$$F = \partial\phi/\partial k. \tag{59}$$

The boundary conditions on F become

$$F(0) = 0, \qquad F'(0) = 0, \qquad F'(\infty) = 0. \tag{60}$$

We write

$$F = \beta G + H. \tag{61}$$

It is easy to see that G and H satisfy exactly the same differential equations as in IVPs (35) and (36). It is also possible to impose the identical missing initial conditions on $G''(0)$ and $H''(0)$, namely that $G''(0) = 1$ and $H''(0) = 0$. Thus we solve the same set of equations for G and H as in the previous section (except of course that the ϕs are now different). However, now β is given by

$$\beta = -H'(\infty)/G'(\infty). \tag{62}$$

The values of G, H and β can be computed using the discretization scheme given in Section 3.1 and F then can be obtained from equation (61). By integrating equation (59), ϕ can be found at the next level of k. Once again it was found convenient to use the second-order Adam–Bashforth method, but it was not necessary to change the step size of k. It was kept uniform at 0·001. Further, the value of β became stationary to within five significant digits at $\zeta=5$ for all values of k. This can possibly be attributed to the fact that the velocity profiles for the type of flow considered in this section do not have an oscillatory character, unlike the case of the previous problem of stagnation point flow.

For the present problem it is possible to show analytically that the solution breaks down for $k=\frac{1}{4}$. Setting $\zeta=0$ in equation (56) and making use of the boundary conditions (57), we get

$$\phi'''(0)=\frac{1}{1-4k}. \tag{63}$$

Thus one should not expect the solutions to exist beyond $k=\frac{1}{4}$. For the two-dimensional problem a similar restriction applies on k.

4.2. Results and discussion

In Figure 4 the velocity profiles are plotted against ζ for various values of k. A perturbation solution was also developed as in Section 3.5. In Table II the values of $\phi''(0)$ are presented using the technique of parameter differentiation and the perturbation method. It can be seen that as k approaches the limiting value $\frac{1}{4}$, the results obtained by the perturbation technique tend to be in

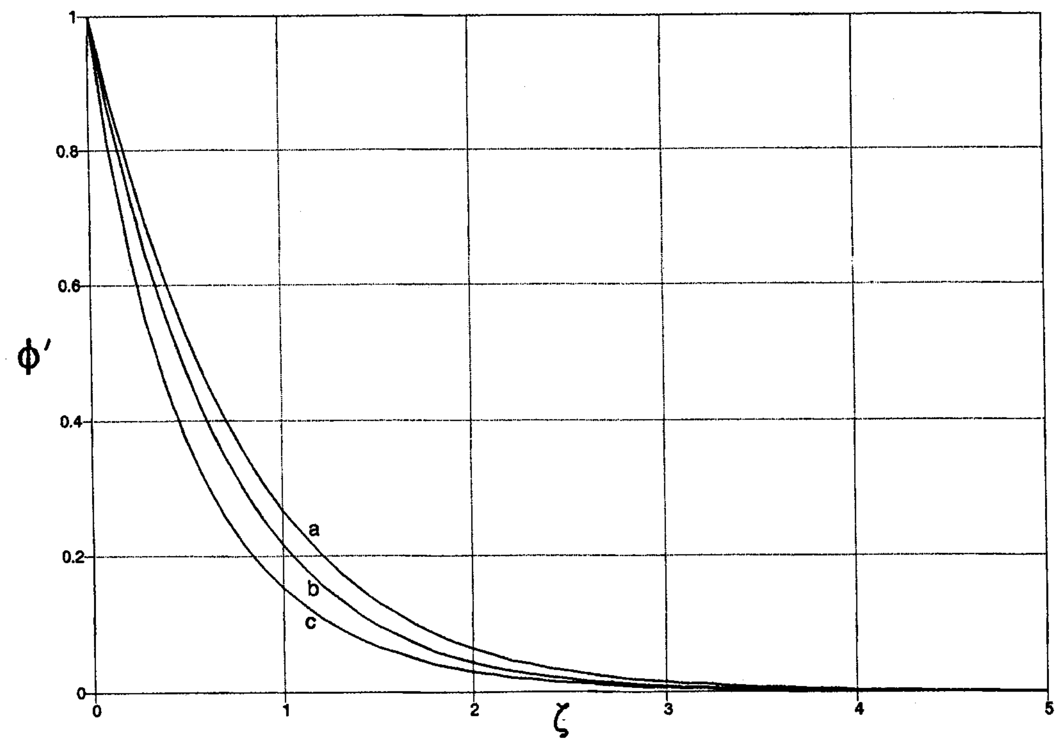

Figure 4. Axisymmetric flow due to radial stretching of a sheet: variation in velocity distribution $\phi'(\zeta)$ with ζ, the similarity variable, for various values of k, the viscoelastic parameter. Curve a, $k=0{\cdot}0$; curve b, $k=0{\cdot}1$; curve c, $k=0{\cdot}2$. No solution exists for $k\geq 0{\cdot}25$

Table II. Variation in $\phi''(0)$ with k for axisymmetric flow due to radial stretching of sheet

k	$\phi''(0)$	
	Exact	Perturbation
0·00	$-1{\cdot}173988$	$-1{\cdot}173988$
0·05	$-1{\cdot}272125$	$-1{\cdot}259434$
0·10	$-1{\cdot}408094$	$-1{\cdot}344880$
0·15	$-1{\cdot}623481$	$-1{\cdot}430325$
0·20	$-2{\cdot}089282$	$-1{\cdot}515771$

gross error compared to the exact results. The perturbation method, as expected, does not anticipate the vertical tangent in the values of $\phi''(0)$ near $k=\frac{1}{4}$. Also, it fails to predict the non-existence of the solution for $k \geq \frac{1}{4}$.

5. CONCLUSIONS

In the present paper we have considered two problems concerning the axisymmetric flow of viscoelastic fluid: (i) flow near a stagnation point and (ii) flow due to radial stretching of a sheet. For both problems, solutions are obtained by the technique of parameter differentiation and the perturbation technique. The exact numerical solutions obtained by the former technique point to the conclusion that whenever possible, one should seek the solution of the full set of equations for viscoelastic fluids, since the solution obtained by the perturbation technique might fail to reveal the vital characteristics of the flow introduced by the viscoelasticity of the fluid. The present paper has added one more algorithm to obtain the desired exact numerical solution. The algorithm has an advantage over those reported earlier by Teipel[11] and Ariel[12] in that it obtains the solutions non-iteratively and is therefore computationally more efficient. It can be used equally effectively to solve flow problems of viscoelastic fluids through porous ducts, between disks, etc., especially when a *family* of solutions is sought for a range of values of the viscoelastic fluid parameter.

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APPENDIX

This appendix is intended to substantiate the numerical results reported in the paper using the parameter differentiation technique. We present one more algorithm which is based on the plausible assumption that ϕ defined by equation (26) and given by BVP (27), (29) is an analytical function of k, the non-Newtonian fluid parameter, at least in the neighbourhood of $k=0$. This implies that ϕ admits a power series expansion in k, which converges within a suitable radius of convergence. We can therefore write

$$\phi = \sum_{i=0}^{\infty} \phi_i k^i, \tag{64}$$

where ϕ_i are functions of ζ.

Note that (64) is a straightforward generalization of the perturbation technique presented in Section 3.5. One of the aims of solving BVP (27), (29) in its entire generality is to obtain its accurate solution. By retaining higher-order terms of k, it is hoped to achieve this aim at least to a certain extent.

For the corresponding problem of two-dimensional stagnation point flow, Beard and Walters,[1] by retaining the first-order terms of k, were able to demonstrate that the velocity in the boundary layer exceeded its value in the mainstream flow. However, they were not able to establish the oscillatory nature of the boundary layer. By taking higher-order terms of k in (64), it may be possible not only to get quantitatively more accurate results but also to obtain qualitatively a more complete picture of the flow characteristics.

Substituting ϕ from (64) into BVP (27), (29) and equating like powers of k, we obtain BVP (54) for ϕ_0 for the Newtonian fluid and the following BVP for ϕ_i ($i \geq 1$):

$$\phi_i''' + 2\phi_0\phi_i'' - 2\phi_0'\phi_i' + 2\phi_0''\phi_i = -\sum_{j=1}^{i-1}(2\phi_{i-j}\phi_j'' - \phi_{i-j}'\phi_j') - 2\sum_{j=0}^{i-1}(\phi_{i-1-j}\phi_j^{\text{iv}} - 2\phi_{i-1-j}'\phi_j'''), \tag{65}$$

$$\phi_i(0)=0, \qquad \phi_i'(0)=0, \qquad \phi_i'(\infty)=0.$$

Let Φ_N denote the partial sum of the series in (64) up to the Nth power of k, i.e.

$$\Phi_N = \sum_{i=0}^{N} \phi_i k^i. \tag{66}$$

For sufficiently small values of k one can expect the sequence $\{\Phi_0, \Phi_1, \Phi_2, \ldots\}$ to converge to the exact solution ϕ. As remarked earlier, the BVPs for ϕ_i are of standard type in which the order of the differential equation is the same as the number of boundary conditions. Moreover, they are linear; therefore their solutions can be obtained non-iteratively by any of the standard techniques. In view of the increasing complexity of BVPs (65), a practical limit must be imposed on i. We chose to limit the solutions for ϕ_i up to $i=3$. The solutions for ϕ_0 and ϕ_1 have already been obtained in Section 3.5. The solutions for ϕ_2 and ϕ_3 were obtained using the shooting method. As we shall see, they have enough information to arrive at some meaningful conclusions.

In Table III the values of Φ_i ($i=0$–3) are presented for $k=0{\cdot}15$. Also given in the table is the exact numerical solution using the parameter differentiation technique. The most interesting aspect of the results is the development of the oscillatory nature of the boundary layer. When only the first-order term of k is taken into consideration, the velocity crosses the mainstream value and approaches it from the top. However, when the second-order term is also included, the velocity crosses back over the mainstream value, giving rise to a wave-like phenomenon. This is further affirmed when the third-order term is included next. The velocity which had become less than the mainstream value for large values of ζ for the second-order solution now again exceeds this value for the third-order solution. The pattern thus seems to be established—addition of each term in equation (64) leads to the crossing by the velocity of its mainstream value and formation or completion of the half-wave front, resulting in oscillations of the boundary layer. This therefore lends weight to the validity of the numerical results obtained in this paper using parameter differentiation.

Another aspect of the generalized perturbation method deserves a mention. Note that as the value of i is increased, Φ_i takes longer distances to approach its asymptotic value of unity. This means that the oscillatory behaviour becomes more pronounced and lasting because of the higher-order terms in (64). However, these terms essentially contribute for larger values of k. Hence as k increases, one can expect oscillations to take place more violently and to extend to

Table III. Variation in ϕ' with ζ for $k=0{\cdot}15$. Φ_i denotes the partial sum of series (64) up to the term k^i

ζ	ϕ'	Φ'_0	Φ'_1	Φ'_2	Φ'_3
0·0	0·000000	0·000000	0·000000	0·000000	0·000000
0·1	0·168812	0·126194	0·148288	0·156982	0·161231
0·2	0·326532	0·242394	0·285831	0·303039	0·311464
0·3	0·472025	0·348632	0·411976	0·437275	0·449704
0·4	0·604171	0·444987	0·526196	0·558886	0·575021
0·5	0·721945	0·531603	0·628131	0·667220	0·686625
0·6	0·824504	0·608710	0·717619	0·761841	0·783935
0·7	0·911269	0·676628	0·794717	0·842571	0·866632
0·8	0·982001	0·735773	0·859717	0·909532	0·934708
0·9	1·036858	0·786655	0·913144	0·963159	0·988491
1·0	1·076421	0·829868	0·955744	1·004200	1·028657
1·1	1·101698	0·866072	0·988457	1·033687	1·056218
1·2	1·114090	0·895977	1·012378	1·052899	1·072494
1·3	1·115332	0·920317	1·028705	1·063302	1·079062
1·4	1·107411	0·939827	1·038689	1·066479	1·077684
1·5	1·092462	0·955221	1·043580	1·064058	1·070229
1·6	1·072665	0·967174	1·044566	1·057627	1·058580
1·7	1·050136	0·976302	1·042740	1·048668	1·044536
1·8	1·026831	0·983158	1·039056	1·038491	1·029731
1·9	1·004466	0·988220	1·034311	1·028182	1·015554
2·0	0·984457	0·991892	1·029136	1·018577	1·003085
2·2	0·955447	0·996345	1·019223	1·003514	0·985882
2·4	0·944166	0·998459	1·011407	0·995072	0·979931
2·6	0·949833	0·999394	1·006141	0·992259	0·982617
2·8	0·967832	0·999778	1·003013	0·992888	0·989285
3·0	0·991512	0·999924	1·001351	0·994872	0·995802
3·2	1·014034	0·999976	1·000555	0·996874	1·000023
3·4	1·029933	0·999993	1·000209	0·998339	1·001767
3·6	1·036146	0·999998	1·000072	0·999219	1·001893
3·8	1·032378	1·000000	1·000023	0·999672	1·001363
4·0	1·020808	1·000000	1·000007	0·999876	1·000783
4·2	1·005263	1·000000	1·000002	0·999958	1·000380
4·4	0·990103	1·000000	1·000000	0·999987	1·000160
4·6	0·979103	1·000000	1·000000	0·999996	1·000059
4·8	0·974600	1·000000	1·000000	0·999999	1·000019
5·0	0·977076	1·000000	1·000000	1·000000	1·000006

larger values of ζ. This is again in line with the results reported in this paper and the earlier works.[11,12]

Useful though the generalized perturbation technique is in the study of the development of the solution, it has got certain limitations. The most serious drawback is of course that no matter how many terms one may take in the expansion (64), the perturbation technique cannot account for a turning point in the solution. This is simply because a polynomial expansion cannot cater for a vertical tangent. Next, even though it is true that, since ϕ_i are universal functions, by a superimposition of their solutions ϕ can be obtained for any 'small' k, the trouble arises when one tries to quantify the smallness of the admissible values of k. A value of 0·2 may be considered small compared to unity, but as the present study shows, this may be too large a value, in fact greater than the critical value of k beyond which no solution exists. Finally, from a practical point of view the computation of ϕ_i beyond a certain value of i requires considerable effort. The series in (64) must therefore be truncated after a few terms. This naturally puts a limit on the value of k if an accurate solution is required. As an example, for the third-order expansion $\phi''(0)$ is given by

$$\phi''(0)=1{\cdot}311\,938+1{\cdot}481\,487\,k+3{\cdot}875\,556\,k^2+12{\cdot}620\,725\,k^3+O(k^4).$$

If an accuracy of nearly 0·5% is required, one can go only as far as $k=0{\cdot}1$. For still higher values of k the above formula gives much worse results, and recourse must be taken to a technique such as parameter differentiation or the iterative algorithms given by Teipel[11] and the present author.[12] For values of k of this size, Teipel's method should not present any difficulties.

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